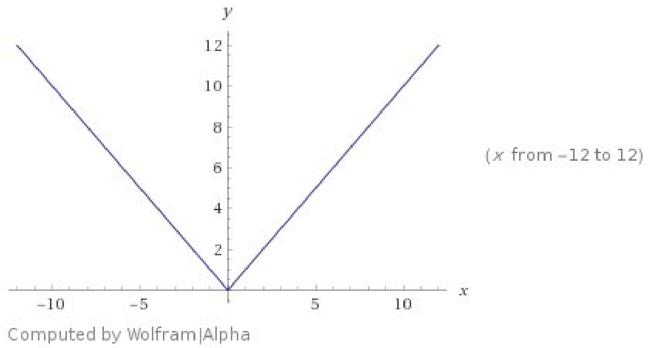
THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010H/I/J University Mathematics 2017-2018
Suggested Solution to Assignment 1

1. $f(4) = \sqrt{4} = 2$, $f(9) = \sqrt{9} = 3$ and $f(16) = \frac{1}{16-9} = \frac{1}{7}$

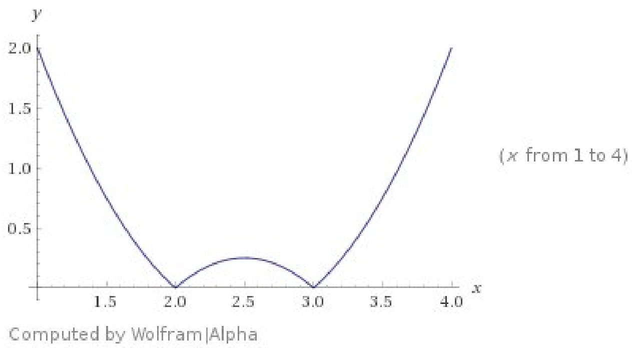
2. (a)

$$f(x) = \begin{cases} x, \text{ if } x \geq 0; \\ -x, \text{ if } x < 0. \end{cases}$$



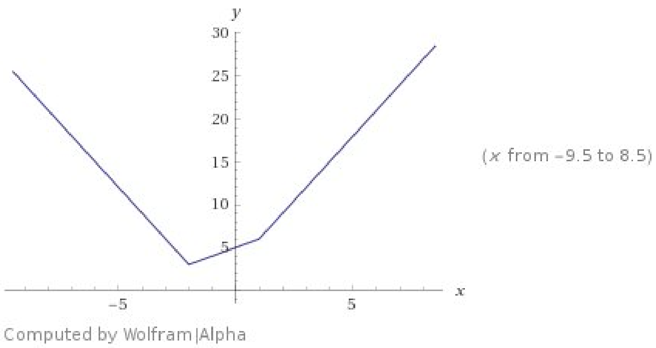
(b)

$$f(x) = \begin{cases} x^2 - 5x + 6, \text{ if } x \geq 3; \\ -x^2 + 5x - 6, \text{ if } 2 \leq x < 3; \\ x^2 - 5x + 6, \text{ if } x \leq 2. \end{cases}$$



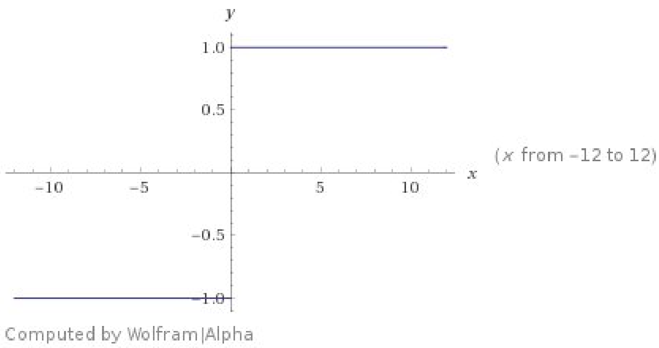
3. (a) $f(x)$ can be written as follow.

$$f(x) = \begin{cases} 2x + 4 + x - 1 = 3x + 3, \text{ if } x \geq 1; \\ 2x + 4 - (x - 1) = x + 5, \text{ if } -2 \leq x < 1; \\ -(2x + 4) - (x - 1) = -3x - 3, \text{ if } x \leq -2. \end{cases}$$



(b) $g(x)$ can be written as follow.

$$g(x) = \begin{cases} -1, \text{ if } x < 0; \\ 0, \text{ if } x = 0; \\ 1, \text{ if } x > 0. \end{cases}$$



4. (a) Denote $g(x) := \frac{1}{2}(f(x) + f(-x))$, then $g(-x) = \frac{1}{2}(f(-x) + f(x)) = g(x)$, which implies that $g(x)$ is a even function;
   Denote $h(x) := \frac{1}{2}(f(x) - f(-x))$, then $h(-x) = \frac{1}{2}(f(-x) - f(x)) = -h(x)$, which implies that $h(x)$ is an odd function;

   (b) By (a), we have that $f(x) = g(x) + h(x)$.

5. (a) Suppose f and g are odd functions i.e. $f(-x) = -f(x)$ and $g(-x) = -g(x)$, then

$$(f \cdot g)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = (f \cdot g)(x)$$

   (b) Suppose f and g are even functions i.e. $f(-x) = f(x)$ and $g(-x) = g(x)$, then

$$(f \cdot g)(-x) = f(-x)g(-x) = f(x)g(x) = (f \cdot g)(x)$$

   (c) Suppose f is an odd function and g is an even function i.e. $f(-x) = -f(x)$ and $g(-x) = g(x)$, then

$$(f \cdot g)(-x) = f(-x)g(-x) = (-f(x))g(x) = -f(x)g(x) = -(f \cdot g)(x)$$

6. (a) Considering the following two formulas

$$\cos 5x \cos 2x + \sin 5x \sin 2x = \cos(5x - 2x) = \cos 3x,$$
$$\cos 5x \cos 2x - \sin 5x \sin 2x = \cos(5x + 2x) = \cos 7x,$$

thus we have that

$$\cos 5x \cos 2x = \frac{1}{2}(\cos 3x + \cos 7x).$$

(b) Using similar method, by

$$\sin 7x \sin 3x + \cos 7x \cos 3x = \cos(7x - 3x) = \cos 4x,$$
$$\sin 7x \sin 3x - \cos 7x \cos 3x = -\cos(7x + 3x) = -\cos 10x$$

thus we have that

$$\sin 7x \sin 3x = \frac{1}{2}(\cos 4x - \cos 10x).$$

(c) By

$$\sin 4x \cos 6x + \cos 4x \sin 6x = \sin 10x,$$
$$\sin 4x \cos 6x - \cos 4x \sin 6x = \sin(-2x) = -\sin 2x,$$

thus we have that

$$\sin 4x \cos 6x = \frac{1}{2}(\sin 10x - \sin 2x).$$

7. (a)

$$\tan x = \frac{\sin x}{\cos x} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2}.$$

(b) Denote $t = \tan\frac{x}{2}$, by (a) we have that

$$\left(\frac{2t}{1-t^2}\right)^2 = \tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{\sin^2 x}{1-\sin^2 x} \implies \sin^2 x = \left(\frac{2t}{1+t^2}\right)^2,$$

and because

$$\sin x \cdot \tan\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2}\tan\frac{x}{2} = 2\sin^2\frac{x}{2} \geq 0;$$

hence we know that $\sin x$ has the same sign with $\tan\frac{x}{2}$, which implies that

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}.$$

Thus,

$$\cos x = \frac{\sin x}{\tan x} = \frac{1-t^2}{1+t^2}.$$

Hence,

$$\frac{1}{2+3\cos x + 4\sin x} = \frac{1}{2 + 3\frac{1-t^2}{1+t^2} + 4\frac{2t}{1+t^2}} = \frac{1+t^2}{5+8t+2t^2+8t^3-3t^4}$$

8. Firstly, we prove a general formula for any $k \in \mathbb{N}$:

$$\sin\big(\theta + (2k-1)\alpha\big) + 2\cos(\theta + 2k\alpha)\cdot\sin\alpha = \sin\big(\theta + (2k+1)\alpha\big).$$

In fact, $\sin\big(\theta+(2k-1)\alpha\big)+2\cos(\theta+2k\alpha)\cdot\sin\alpha$

$$
\begin{aligned}
&=\sin\big(\theta+(2k-1)\alpha\big)+2\sin\alpha\cdot\big[\cos\big(\theta+(2k-1)\alpha\big)\cos\alpha-\sin\big(\theta+(2k-1)\alpha\big)\sin\alpha\big]\\
&=\sin\big(\theta+(2k-1)\alpha\big)\cdot(1-2\sin^2\alpha)+\cos\big(\theta+(2k-1)\alpha\big)\sin 2\alpha\\
&=\sin\big(\theta+(2k-1)\alpha\big)\cos 2\alpha+\cos\big(\theta+(2k-1)\alpha\big)\sin 2\alpha=\sin\big(\theta+(2k+1)\alpha\big).
\end{aligned}
$$

Thus, we have that

$$
\begin{aligned}
&\sin(\theta-\alpha)+2\sin\alpha\big[\cos\theta+\cos(\theta+2\alpha)+\cdots+\cos(\theta+8\alpha)\big]\\
=&\sin(\theta+\alpha)+2\sin\alpha\big[\cos(\theta+2\alpha)+\cdots+\cos(\theta+8\alpha)\big]\\
=&\cdots=\sin(\theta+7\alpha)+\cos(\theta+8\alpha)\cdot\sin\alpha=\sin(\theta+9\alpha).
\end{aligned}
$$

Taking $\alpha=\frac{\pi}{5}$ into the formula we proved, then by $\sin\frac{\pi}{5}\neq 0$, and

$$\sin(\theta+\frac{9\pi}{5})=\sin(\theta+\frac{9\pi}{5}-2\pi)=-\sin(\theta-\frac{\pi}{5}),$$

we have that

$$\cos\theta+\cos(\theta+\frac{2\pi}{5})+\cos(\theta+\frac{4\pi}{5})+\cos(\theta+\frac{6\pi}{5})+\cos(\theta+\frac{8\pi}{5})=0.$$

9. (a) Using mathematical induction, as $x_1\leq 4$, and we assume that $x_k\leq 4,\forall 1\leq k\leq n$, then $x_{n+1}=3+\frac{1}{16}x_n^2\leq 3+\frac{1}{16}\cdot 16=4.$.

   (b) We also use mathematical induction:

$$
\begin{aligned}
&x_2=3+\frac{1}{16}x_1^2=3+\frac{9}{16}>x_1 \implies x_2-x_1\geq 0,\\
&x_{n+1}=3+\frac{1}{16}x_n^2 \implies x_{n+1}-x_n=\frac{1}{16}(x_n^2-x_{n-1}^2)=\frac{1}{16}(x_n+x_{n-1})(x_n-x_{n-1}),
\end{aligned}
$$

   by the fact that all $x_n>0,\forall n\geq 1$, thus $x_{n+1}-x_n$ and x_n-x_{n-1} have same sign. Thus by induction, $x_{n+1}\geq x_n,\forall n\geq 1$.

10. (a)

$$
\begin{aligned}
(x+y)^{n+1}&=(x+y)\sum_{r=0}^{n}C_r^n x^r y^{n-r}=\sum_{r=0}^{n}C_r^n(x^{r+1}y^{n-r}+x^r y^{n+1-r})\\
&=\sum_{l=1}^{n}C_{l-1}^n x^l y^{n+1-l}+\sum_{r=1}^{n}C_r^n x^r y^{n+1-r}+x^{n+1}+y^{n+1}\\
&=\sum_{r=1}^{n}(C_{r-1}^n+C_r^n)x^r y^{n+1-r}+x^{n+1}+y^{n+1}
\end{aligned}
$$

Because

$$
\begin{aligned}
C_{r-1}^n+C_r^n&=\frac{n!}{(r-1)!(n+1-r)!}+\frac{n!}{r!(n-r)!}=\frac{n!}{(r-1)!(n-r)!}\cdot(\frac{1}{n+1-r}+\frac{1}{r})\\
&=\frac{n!}{(r-1)!(n-r)!}\cdot\frac{n+1}{(n+1-r)\cdot r}=\frac{(n+1)!}{(n+1-r)!r!}=C_r^{n+1},
\end{aligned}
$$

thus $(x+y)^{n+1}=\sum_{r=0}^{n+1}C_r^{n+1}x^r y^{n+1-r}$.

(b)

$$\begin{aligned}(3x-2)^5 &= (3x)^5 + C_4^5 \cdot (3x)^4 \cdot (-2) + C_3^5 \cdot (3x)^3 \cdot (-2)^2 + C_2^5 \cdot (3x)^2 \cdot (-2)^3 + C_1^5 \cdot (3x) \cdot (-2)^4 + (-2)^5 \\ &= 243x^5 - 810x^4 + 1080x^3 - 720x^2 + 240x - 32.\end{aligned}$$

11. By Problem 10,

$$\begin{aligned}(x+h)^n - x^n &= \sum_{r=0}^{n} C_r^n x^{n-r} h^r - x^n = \sum_{r=1}^{n} C_r^n x^{n-r} h^r \\ \Longrightarrow \frac{(x+h)^n - x^n}{h} &= \sum_{r=1}^{n} C_r^n x^{n-r} h^{r-1}.\end{aligned}$$